Portfolio Optimization in an Upside Potential and Downside Risk Framework.

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Humans have always engaged in risk-averse and risk-seeking behavior. As a result, reverse S-shaped utility functions have been utilized to describe this human investment behavior since Friedman and Savage (1948) and Markowitz (1952). Fishburn (1977) made this approach operational with the Lower Partial Moment, LPM(a,t), model which detailed risk-seeking and risk-averse behavior below a minimum target return. However, the Fishburn utility measures have drawn criticism since they assume linear utility above the target return. Recently, the Upper Partial Moment/Lower Partial Moment (UPM/LPM) has been put forward as a solution to this problem. This model can explain risk-seeking and risk-averse behavior above as well as below the target return. This paper develops a general UPM/LPM model that may be used to explain several cases of investor behavior that have appeared in the literature.
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Introduction

There is a lot of evidence indicating that investors are more sensitive to losses than to gains.¹ This introduces a discontinuous change in the shape of the investor’s utility function at some target return and plays a role in Prospect Theory developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1991). However, there is evidence that investors are not risk-averse throughout the range of returns and will exhibit risk-seeking behavior in special situations. Friedman and Savage (1948) and Markowitz (1952) argue the willingness to purchase both insurance and lottery tickets implies reverse S-shaped (both concave and convex) utility functions. A reverse S-shaped utility function provides an explanation for investors engaging in risk-averse behavior for losses and risk-seeking behavior for gains.

Fishburn (1977) proposed the Lower Partial Moment (LPM) \(a,\tau\) model to explain risk-seeking and risk-averse behavior below a target return \(\tau\). Investor behavior is explained through a coefficient \(a\) as \(a<1\) is risk-seeking behavior and \(a>1\) is risk-averse behavior, thus the LPM \(a,\tau\) model. The LPM \(a,\tau\) model proved to be a very useful risk measure because of its flexibility in capturing investor behavior (Nawrocki, 1999). However, it was not immune to criticism. Kaplan and Siegel (1994a, 1994b) zeroed in on its characteristic of a linear utility function above the target return which assumes that the investor is risk-neutral to all above-target returns. A recent paper by Post and von Vliet (2002) found evidence that while investors are risk-averse to below-target returns, they are risk-seeking above the target return.

¹ See Nawrocki (1999) for an overview of investor attitudes towards downside risk.
return. In order to apply more realistic behavior to above-target returns, Sortino, van der Meer and Plantinga (1999) proposed a performance measure, the UPM/LPM ratio. In this paper, we develop a general UPM/LPM portfolio optimization model and demonstrate how it may be used to explain several different cases of investor behavior. The first section of the paper describes the advantages and disadvantages of mean-variance and mean-downside risk (LPM) portfolio models and the need for a general UPM/LPM portfolio model. The second section develops and presents the general UPM/LPM model while the third section presents four different cases of investor utility in order to demonstrate the usefulness of the general model. Finally, conclusions will be offered.

I. The Need for a General UPM/LPM Portfolio Model

The starting point in the discussion is what we should expect from a new portfolio model in general along with a brief summary of advantages and disadvantages of the most commonly used portfolio models: mean-variance and mean-downside risk models.

A. Pros and Cons of the Mean-Variance Optimization Model.

The most desirable property of variance is that it captures returns for the whole probability distribution. However, the upper part of the distribution is minimized in the optimization algorithm. This is a problem as there are desirable returns higher than the mean. The mean-variance model also does not capture higher moments of return distribution such as skewness and kurtosis (fat tails). Variance as a risk measure assumes only one type of risk preference while investors typically exhibit a wide range of risk preferences. In addition, the covariance assumes symmetric correlation between assets, but in real markets the correlation on the downside portion of the distribution is significantly higher than the correlation on the upside.
For a better understanding of mean-variance model, we depict the utility function resulting from the consistency of the mean-variance optimization and Bernoulli’s expected utility criterion. Such a utility function is a quadratic function:

\[ U = r - kr^2, \]

where \( k \) is investor's marginal rate of substitution of expected return for variance.

\[ U(r) \]

\[ \frac{1}{2k} \]

\[ r \]

Figure 1: Quadratic utility function. Source: Markowitz (1959), p. 289.

B. Pros and Cons of Optimization Using a Mean-Downside Risk Model

The major advantage of downside risk in comparison with variance is that by its minimization, return deviations on the upside are not minimized because the downside risk captures only downside deviations from the benchmark or target return. However, the model does not capture different investor preferences on the upside deviations from the benchmark. More precisely, it implies risk neutrality, because returns in the upper part of the distribution influence portfolio allocation only as an input to computing and maximizing the mean. Downside risk expresses a bright spectrum of risk preferences in the downside part of return distribution. The model takes into account investors’ skewness and kurtosis preferences. Therefore, an asymmetric and leptokurtic return distribution influences portfolio allocation.

\[ ^2 \text{The target return or “safety first” return was first described by Roy(1952). Markowitz(1959) integrated the target return into the target semivariance measure which is the LPM degree 2 model (a=2).} \]
The utility function implied by mean-downside risk optimization framework under the assumption of consistency with Bernoulli’s expected utility criterion is as follows:

\[ U(r) = r, \text{ for all } r \geq \tau \]  
\[ U(r) = r - k(\tau - r)^a, \text{ for all } r \leq \tau \]  

\[ 0 \quad \frac{1}{2} \quad 1 \quad 2 \quad 4 \]  
\[ U(r) \quad \tau \]

Figure 2: Types of utility functions in mean-downside risk optimization consistent with Bernoulli’s expected utility criterion. For example, \( a=0, \frac{1}{2}, 1, 2, 4 \), but the consistency is valid for all \( a \leq 0, \infty \); \( k=1 \) for target rate \( \tau \). Source: Fishburn (1977), p.121.

The below-the-benchmark part of the utility function can express various risk preferences, such as risk aversion for \( a>1 \), risk neutrality for \( a=1 \), and risk seeking for \( 0<a<1 \). Risk neutrality above the benchmark return implies linear utility function above the benchmark, which is the most common criticism of portfolio optimization based on mean and downside risk.

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Fishburn (1977) described utility functions implied by mean-downside risk decision criteria: “The general impression obtained from these studies is that most individuals in investment contexts do indeed exhibit a target return—which can be above, at or below the point of no gain and no loss—at which there is a pronounced change in the shape of their utility functions, and that the utility function below the target can give a reasonably good fit to most of these curves in the below–target region. However, the linearity of the utility function above the target holds only in a limited number of cases for returns above-target.”
C. Requirements for a General Portfolio Model

A logical progression would be a combination that provides the advantages of both models and at the same time eliminates the disadvantages of both models. Consequently, we want to construct a portfolio model where the investor can express his/her arbitrary preferences while using the whole return distribution. In the downside part of distribution, variable risk preferences should be expressed using downside risk. The upside deviations from the benchmark should not be minimized as in case of mean-variance or considered risk neutral as in the mean-downside risk model. The benchmark, from which downside deviations are measured, should be a constant value as in risk, and should represent the minimum target return that must be earned in order to accomplish the policy goals of the investor.

This new portfolio model should also take into account higher moments of return distribution such as skewness and kurtosis. In the following sections we will try to develop applicable portfolio model that will fulfill these requirements.

II. A Possible Solution: The Upside Potential-Downside Risk Portfolio Model

The upside potential-downside risk (UPM/LPM) model may be formulated as follows:

Maximize

\[ E(UPM_{\mu}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j E(UPM_{\mu}) \]

where

\[ E(UPM_{\mu}) = 1/K \sum_{t=1}^{K} [Max\{0, (R_{t} - \tau)\}]^{\tau} (R_{t} - \tau) \]

Minimize

\[ E(LPM_{\text{portf}}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j E(LPM_{\text{portf}}) \]

where
\[
E(\text{CLPM}_{ij}) = 1 / K \sum_{i=1}^{K} \left[ \text{Max} \{0, (\tau - R_{ij})\} \right]^{a-1} (\tau - R_{ij})
\]  
(6)

subject to:
\[
\sum_{i=1}^{n} w_i = 1
\]  
(7)

The above multi-objective optimization problem may be alternatively formulated as a
minimization of downside risk (LPM) for a certain target “b“ level of upside potential (UPM-
upside partial moment).

**Minimize**

\[
E(\text{LPM}_{port}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j E(\text{CLPM}_{ij})
\]  
(8)

subject to:
\[
b = E(\text{UPM}_{p}) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j E(\text{UPM}_{ij})
\]

where

\[
E(\text{UPM}_{ij}) = 1 / K \sum_{i=1}^{K} \left[ \text{Max} \{0, (R_{ij} - \tau)\} \right]^{c-1} (R_{ij} - \tau)
\]  
(9)

\[
\sum_{j=1}^{n} w_j = 1
\]  
(10)

In this portfolio model, the desirable property of mean-downside risk model, i.e.
minimizing deviations only below the target return, remains unchanged. So we are not
minimizing the returns above the target. With the exponent a<1 we can express risk seeking,
a=1 risk neutrality, and a>1 risk aversion behavior on the part of the investor. Risk aversion
means the further returns fall below the target return, the more we dislike them. On the other
hand, risk seeking behavior means that the further returns fall below the target return, the
more we prefer them.
The major difference from the traditional mean-downside risk model is the replacement of the expected portfolio return maximization with the maximization of the expected upside return potential (UPM-upside partial moment). The UPM measure captures the upside return deviations from benchmark. Therefore, the expected upside partial moment $E(UPM)^{1/c}$ of a portfolio can be interpreted as an expected return potential of portfolio relative to a benchmark.

$$E(UPM)^{1/c} = \left[ \frac{1}{K} \sum_{k=1}^{K} \max[(R_k - \tau);0] \right]^{1/c}$$ (11)

Similar to the downside risk LPM calculation, the UPM could also be expressed as an expected upside deviation from benchmark multiplied by the related probability:

$$UPM = \frac{1}{K} \sum_{k=1}^{K} [(R_k - \tau)^c |R_k > \tau| \cdot P(R_k > \tau)$$ (12)

The UPM contains important information about how often and how far investor wishes to exceed the benchmark, which the mean return ignores.

We do not consider the upside deviation from benchmark to be risk, therefore, we label it as an upside return potential. As in the LPM calculation, different exponents represent different investor behaviors: potential seeking, potential neutrality or potential aversion above the benchmark return. Potential seeking means the higher the returns above the target return, the happier the investor. The potential aversion describes a rather conservative strategy on the upside. Because of the maximization of the UPM, the exponent $c<1$ represents potential aversion, $c=1$ potential neutrality, and $c>1$ potential seeking. Hence, the often criticized utility neutrality above the benchmark that is inherent in the mean-downside risk model and the potential aversion inherent in the mean-variance model is eliminated. The exponent “$c$” does not have to be the same value as the penalizing exponent “$a$”; so we can combine different strategies, for example, a risk aversion $a=4$ and a potential aversion $c=0.5$ might represent a conservative investor.

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4 For example, such a strategy could utilize a short call or a short put and their dynamic replication with stock and bonds.
Most investors consider protection against losses as more important than exposure to gain, so the a-exponent will usually be higher than the c-exponent.

The following example depicts the difference between mean and UPM. Table 1 provides returns for assets x1 and x2, while table 2 lists the values for the means and UPM values for x1 and x2. The means of the assets are identical; however, the UPM values differ according to the investor’s aversion towards upside potential. The potential upside averse investor (c<1) prefers the more conservative asset x1 while the potential seeking investor (c>1) would prefer asset x2.

Table 1: Example Data for x1 and x2.

<table>
<thead>
<tr>
<th>x1</th>
<th>x2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4.5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

This example does not take into account asset risk; so if we prefer UPM of one asset, we do not have to prefer it in terms of its risk-return trade off.
Table 2: Mean and UPM Values for Example

<table>
<thead>
<tr>
<th></th>
<th>x1</th>
<th>x2</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>3.08</td>
<td>3.08</td>
</tr>
<tr>
<td>UPM (c=0.25; t=2)</td>
<td>0.80</td>
<td>0.65</td>
</tr>
<tr>
<td>UPM (c=0.5; t=2)</td>
<td>0.87</td>
<td>0.86</td>
</tr>
<tr>
<td>UPM (c=1; t=2)</td>
<td>1.08</td>
<td>1.54</td>
</tr>
<tr>
<td>UPM (c=2; t=2)</td>
<td>2.25</td>
<td>5.35</td>
</tr>
<tr>
<td>UPM (c=3; t=2)</td>
<td>6.58</td>
<td>19.64</td>
</tr>
</tbody>
</table>

A. Utility Functions Using UPM/LPM Analysis

At this point, it should be clear that this portfolio model implies a bright spectrum of utility functions (See Figure 3). The variability of the below benchmark returns is similar to Fishburn’s utility functions employing the LPM measure. However, the upper part of the return distribution exhibits variable investor behavior and is not limited to the potential neutrality imposed by the Fishburn’s utility functions. Combining different exponents a and c, we can describe additional types of investor behavior. There are different non-linear and linear utility functions. The reverse S-shaped utility function described in general for a>1 and c>1 is consistent with insurance against losses and taking bets for gains (in the figure a=c=2, 3 or 4). In Figure 4, we see that a=2, c=0.5 is one of combinations of a, c, and t that can approximate the traditional quadratic utility curve. This approximated utility differs only for greater values than the zenith of the (μ, σ)- utility function \( r \leq 1/2k \). Behind this point the (μ, σ)- utility decreases with increasing final wealth, which is irrational. However, with the utility function based on the (UPM, LPM) model, investor utility increases with the increasing final wealth.

The utility functions of prospect theory that are used in behavioral finance (Tversky [1995]) will be the S-shaped utility functions for 0<a<1 and 0<c<1 (In Figure 3, a=c=0.5 or
These utility functions capture the investors’ tendency to make risk-averse choices relative to UPM and risk-seeking choices relative to LPM. Investors are very risk-averse to small losses but will take on investments with a small chance of very large losses.

Also, for individuals with the potential and risk seeking behaviour, the \((UPM, LPM)\) portfolio model can be applied. Then, \(0 < a < 1\) and \(c > 1\) imply a convex utility function.

In addition, risk neutrality \((a = 1)\) in combination with potential aversion or potential seeking, i.e. linear gain function and concave or convex loss function, can be expressed. Also, the upper potential neutrality \((c = 1)\) in combination with downside risk-aversion or risk-seeking, which implies a linear utility function above the target and concave or convex below the target, is allowed. Linear gain and loss function can be also assumed by \(a = 1\) and \(c = 1\), which means that the gains and losses are evaluated proportionally to their extension.

The benchmark return should be the minimum return required to accomplish the policy goals of the investor (i.e., the return necessary to cover the liabilities of the investor).

The two objectives of maximizing the return and minimizing the risk can be viewed either as a multi-objective optimization problem, or the objectives can be combined using a utility function. Then, the portfolio’s expected utility can be interpreted as a risk-adjusted expected return potential, since it is computed by subtracting a risk penalty from the expected return.\(^6\)

\[E(U) = \text{expected return} - h \cdot \text{expected risk}\]

To obtain the efficient portfolio, expected utility has to be maximized for a given parameter \(h > 0\), which represents the investor’s risk tolerance, i.e. investor's marginal rate of substitution of expected value for expected risk. Computing efficient portfolios for different values of the \(h\)-parameter, we can generate an efficient frontier.

In case of expected return potential (UPM) and downside risk LPM, expected utility, as a risk-adjusted expected return potential, is computed by subtracting a downside risk penalty

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\(^6\)For a proof, see: Markowitz (1959), p.287. See, for example, Womersley and Lau (1996). However, note that in this paper the expected return is replaced by the expected return potential (UPM).
from the expected return potential. In order to obtain efficient portfolios, we have to maximize:

Maximize:

\[ E(U) = \text{expected return potential} - h \cdot \text{expected downside risk}, \text{ or} \]

\[ E(U) = E(UPM) - h \cdot E(LPM) \quad (13) \]

This expected utility function is the expected value of the utility function:

\[ U = \max [(R-\tau); 0] \quad (14) \]

Where,

\[ U(r) = (\tau - r)^c, \quad \text{for all } r \geq \tau \text{ and} \]

\[ U(r) = -h \cdot (\tau - r)^a, \quad \text{for all } r \leq \tau \text{ and } h > 0. \quad (16) \]

![Utility functions of UPM-LPM portfolio model for different exponents (t=0; h=1).](image)

**Figure 3:** Utility functions of UPM-LPM portfolio model for different exponents (t=0; h=1).
B. Estimation of the Amount of Downside Risk Aversion and Upside Potential Exposure

We can approximately estimate the investor risk behavior exponent “a” using the following methodology. We always compare two alternatives with the same UPM, but with different LPM values. In the first example, we can lose A-amount with p–probability. In the second example, we can lose B–amount with the same probability p, but for two states. Using the exponent a=1, i.e. risk neutrality, we would be indifferent between these two alternatives for the B-amount equal to A/2.

\[
p \cdot A^l = p \cdot B^l + p \cdot B^l
\]

\[
p \cdot A^l = p \cdot \left( \frac{A}{2} \right)^l + p \cdot \left( \frac{A}{2} \right)^l
\]

Figure 4: Quadratic utility function for a=2, c=0.5

If we prefer the second alternative, we would have a higher grade of risk aversion because this second alternative has a lower total loss for all cases of higher risk aversion, or a>1. To compute the amount of risk aversion, we will have to compare the same two alternatives with a higher degree of risk aversion, for example a=2.
\[ p \cdot A^2 = p \cdot B^2 + p \cdot B^2 \]
\[ p \cdot A^2 = p \cdot \left( \frac{A}{\sqrt{2}} \right)^2 + p \cdot \left( \frac{A}{\sqrt{2}} \right)^2 \]  

(18)

If we have the degree of risk aversion \( a = 2 \), we would be indifferent between the loss of \( A \)-amount with \( p \)-probability and the loss of \( A/\sqrt{2} \) amount with the same probability \( p \). If we prefer the second alternative, we have a higher degree of risk aversion than \( a = 2 \). So we have to repeat the comparison of these two alternatives for higher and higher degrees of \( a \). For example, for the case \( \text{“} a \text{”} = 3 \), we are indifferent between the alternatives, where the loss of \( B \) in the second alternative is equal to \( A/\sqrt{2} \). If we prefer the second alternative, then we have even a higher degree of risk aversion, and we have to repeat the comparison until we find indifferent alternatives.

In case of a risk seeking investor, we would prefer by the first alternative when \( a = 1 \). From there, we have to reduce the degree of risk seeking behavior (\( a < 1 \)) until we reach indifference.

For the degree of return exposure in the upper part of distribution \( \text{“} c \text{”} \), we proceed the same way, however, potential seeking behavior is \( c > 1 \) and the conservative strategy on the upside is \( 0 < c < 1 \).

C. Summary of Assets Used in Four UPM-LPM Utility Cases

The next section of the paper will present four utility cases to illustrate the use of the UPM/LPM model. To help present these cases, 12 assets were utilized. Their summary statistics are presented in Table 3. As can be seen, the assets present a wide spectrum of high and low returns, high and low standard deviations, and positive and negative skewness in order demonstrate the properties of the UPM/LPM model.
<table>
<thead>
<tr>
<th>Asset</th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.185</td>
<td>0.382</td>
<td>0.166</td>
</tr>
<tr>
<td>2</td>
<td>0.079</td>
<td>0.230</td>
<td>-0.060</td>
</tr>
<tr>
<td>3</td>
<td>0.215</td>
<td>0.462</td>
<td>0.380</td>
</tr>
<tr>
<td>4</td>
<td>0.175</td>
<td>0.351</td>
<td>0.413</td>
</tr>
<tr>
<td>5</td>
<td>0.179</td>
<td>0.195</td>
<td>-2.937</td>
</tr>
<tr>
<td>6</td>
<td>0.095</td>
<td>0.115</td>
<td>-2.916</td>
</tr>
<tr>
<td>7</td>
<td>0.179</td>
<td>0.217</td>
<td>-2.050</td>
</tr>
<tr>
<td>8</td>
<td>0.146</td>
<td>0.160</td>
<td>-3.074</td>
</tr>
<tr>
<td>9</td>
<td>0.052</td>
<td>0.241</td>
<td>1.372</td>
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<td>10</td>
<td>0.022</td>
<td>0.147</td>
<td>0.913</td>
</tr>
<tr>
<td>11</td>
<td>0.065</td>
<td>0.296</td>
<td>1.294</td>
</tr>
<tr>
<td>12</td>
<td>0.059</td>
<td>0.239</td>
<td>1.518</td>
</tr>
</tbody>
</table>

Table 3 – Summary Statistics for 12 Assets Used in Four Utility Cases

### III. Four Utility Cases

**Case 1: Downside Risk Aversion (\( a=2 \)) and Upside Potential Seeking (\( c=3 \)) Investment Strategy**

This is probably a very common case where investors wish to reduce downside risk while at the same time preserving as much of the upside returns as economically feasible. This means that the investor is risk averse below the target return and upside potential seeking above the target. The utility function expressing such preferences is \( \mathcal{Z} \)-formed (reverse S-shaped), which means that it is concave below the target return and convex above the target.

In the utility function where return deviations from some reference point are evaluated by some exponent, risk aversion is represented by an exponent \( a \geq 1 \), and upside seeking is represented by \( c \geq 1 \). In this example, we assume that the degree of risk aversion is identified \( a = 2 \), and degree of upside potential seeking is \( c = 3 \). The minimum target return is set equal to the risk free return of 3%. The utility function \( U(r) \) is defined as:

\[
U(r) = \begin{cases} 
(r-0.03)^3, & \text{for all } r \geq 0.03, \text{ and} \\
-h \cdot (0.03-r)^2, & \text{for all } r < 0.03 \text{ or}
\end{cases}
\]

\[
U(r) = \max[(r-0.03) ;0]^3 -h \cdot \max[(0.03-r) ;0]^2
\]
Figure 5: Utility Function of Downside Risk Averse \((a=2)\) and Upside Potential Seeking \((c=3)\) Investment Strategy.

**a. Efficient Frontiers for \(a=2, c=3\)**

The efficient frontier of portfolios with maximal expected utility we obtain by varying the slope - \(h\)-parameter – by the maximization of the \(EU(r)\). The efficient frontiers are computed using the \((UPM_{3,3}, LPM_{2,3})\), \((\mu, LPM_{2,3})\), and \((\mu, \sigma)\) portfolio models. Using the \((UPM_{3,3}, LPM_{2,3})\) coordinate system (Figure 18), we see that the UPM/LPM frontier is dominate while the \((\mu, \sigma)\) frontier is only partly concave.

Depicting efficient frontiers in the \((\mu, \sigma)\) coordinate system (Figure 19), we see that the \((UPM_{3,3}, LPM_{2,3})\) optimized portfolios are shifted to the right, so we can expect for the same level of portfolio return as the \((\mu, LPM_{2,3})\) portfolio, but with higher standard deviation. The \((UPM_{3,3}, LPM_{2,3})\) efficient frontier is only partially concave in the \((\mu, \sigma)\)- coordinate system.
b. The Probability Distributions for Optimal Portfolios (a=2) and (c=3)

The return probability distributions of all portfolio types with maximal expected utility shows truncated exposure to below-target returns, and increased exposure to high returns above the target as the utility function requires (Figure 20).
The return distributions of portfolios optimized with other models are considerably different. The \((UPM_{3,3}, LPM_{2,3})\) optimal portfolios for the same level of shortfall risk exhibit an increase in the probability of the highest returns in comparison with the \((\mu, \sigma)\) and \((\mu, LPM_{2,3})\) optimized portfolios. This modification of the probability distribution corresponds with assumed investor’s preferences: increased exposure to the highest returns and insurance against shortfall. The increase in the probability of the highest returns results from the optimization with the \(UPM_{3,3}\) measure. As the exponent \(c\) is higher than one, high upside deviations from the target return are relatively more desirable than lower ones. The truncated probability on the downside in the \((UPM_{3,3}, LPM_{2,3})\) and the \((\mu, LPM_{2,3})\) optimized portfolios causes the penalizing exponent \(a = 2\) in \(LPM\) which makes the large downside deviations relatively less desirable than the smaller deviations. The increased exposure to the highest returns makes the return distribution of the \((UPM_{3,3}, LPM_{2,3})\) portfolios wider, which in turn increases the standard deviation and shifts the efficient frontier of these portfolios to the right in the \((\mu, \sigma)\) coordinate system.
The truncated upside potential in the \((\mu, \sigma)\) optimized portfolios reflects the penalization of both (upside and downside) return deviations from the mean whenever the standard deviation is minimized.

The high returns of the \((\mu, LPM_{2,3})\) optimized portfolios are closer to the \((\mu, \sigma)\) portfolios, because the upside return deviations from the target are not penalized. As this model assumes neutrality of preferences above the target, its portfolios are not aggressive towards the upside potential. Therefore, the occurrence of the highest returns is much lower than in the \((UPM_{3,3}, LPM_{2,3})\) portfolios.

**Case 2: Downside Risk Aversion \(a=2\) and Upside Potential Aversion \(c=0.5\) Investment Strategy.**

As investors differ in their preferences, it is possible that many of them are downside risk averse and upside potential averse. Such a preference corresponds with a conservative investment strategy. This strategy would try to concentrate returns towards some target return \(t\). The implied utility function is concave as assumed in the classical theory of expected utility. In the following example, we assume that the degree of upside potential aversion is \(c = 0.5\) and the degree of downside risk aversion is \(a = 2^7\). The minimum target return \(t\) is set equal to risk free return of 3%.

Assume the following utility function:

\[
U(r) = \max[(r-0.03) ; 0]^{0.5} - h \cdot \max[(0.03-r) ; 0]^2
\]

(19)

\(^7\) See the method for estimation of degree of risk aversion and potential seeking presented in Section IIB of this paper.
The general $UPM$ and $LPM$ measures we define according to the assumed utility function as $UPM_{c,t}$ and $LPM_{a,t}$ or $UPM0.5;3$, and $LPM2;3$. The first part of the equation of the expected utility function corresponds with the applied $UPM$ and the second part with the $LPM$ measure.

**a. Efficient Frontiers for $a=2$, $c=0.5$.**

We obtain the efficient frontier of portfolios with maximal expected utility by varying the slope - $h$-parameter – in the $EU(r)$ optimization formulation. The data from Table 3 is used to calculate the efficient portfolios. The $(UPM_{0.5;3}, LPM_{2;3})$, $(\mu,\sigma)$, and $(\mu,LPM_{2;3})$ portfolios provide roughly the same efficient frontiers except in the higher risk area. An alternative strategy, $(UPM_{3;3}, LPM_{2;3})$, where the investor has strong emphasis on seeking upside potential generates portfolios with significantly higher risk. However, it should be noted that the graph uses a $(UPM_{0.5;3}, LPM_{2;3})$ coordinate system.
b. The Probability Distributions for Portfolios Generated Using $a=2$, $c=0.5$.

A number of return distribution graphs of the portfolios were generated. Generally, the graphs indicate the $(UPM_{0.5,3}, LPM_{2,3})$ methodology generated portfolios where the exposure to low and high returns is truncated which corresponds with the assumed preferences of an investor ($a=2$, $c=0.5$). A representative graph is presented in Figure 8.
Case 3: Downside Risk Seeking (a=0.9) and Upside Potential Aversion (c=0.5) Investment Strategy.

If the investor’s main concern is not to fall short but without regard of the amount, and to exceed the target return without regard of the amount, then the appropriate utility function is risk seeking below the target, and upside potential averse above the target. Such preferences are not unusual as confirmed by many experimental studies\(^8\). In addition, there is a strong correspondence with utility functions contained in prospect theory. (Tversky[1995]). These preferences indicate a tendency by investors to make risk-averse choices in gains and risk-seeking choices in losses. Such investors are very risk-averse for small losses but will take on investments with small probabilities of very large losses.

\(^8\) Swalm (1966) found that the predominant pattern below t=0 is a slight amount of convexity, so that a<1 is descriptive for most of the utility curves found in the study.
Assume the degree of the downside risk seeking to be slightly below risk neutrality \( a = 0.9 \), as this is the most common finding in the Swalm’s (1966) experimental study. The degree of upside potential aversion is assumed to be \( c = 0.5 \), and the minimum target return is unchanged at 3%.

This implies the following utility function:

\[
U(r) = \max[(r-0.03) ; 0]^0.5 - h \cdot \max[(0.03-r) ; 0]^0.9 \tag{20}
\]

![Utility Function of the Upside Potential Averse (c=0.5) and Downside Risk Seeking (a=0.9) Investor.](image)

The resulting utility function is approximately linear below the target and concave above the target.

**a. Efficient Frontiers for a=0.9, c=0.5.**

We have to compute new (\( UPM_{0.5,3} \), \( LPM_{0.9,3} \)) and (\( \mu, LPM_{0.9,3} \)) portfolios while the (\( \mu, \sigma \)) portfolios remain the same.
Again, the \((UPM_{0.5;3}, LPM_{0.9;3})\) efficient frontier dominates when using the \((UPM_{0.5;3}, LPM_{0.9;3})\) axis units. The other efficient frontiers differ mostly for high values of downside risk.

Figure 13: \((UPM_{0.5;3}, LPM_{0.9;3}), (\mu, LPM_{0.9;3}), (\mu, \sigma)\) and \((UPM_{3;3}, LPM_{0.9;3})\) efficient frontiers in a \((UPM_{0.5;3}, LPM_{0.9;3})\) framework.\(^9\)

Figure 14: \((UPM_{0.5;3}, LPM_{0.9;3}), (UPM_{3;3}, LPM_{0.9;3}), (\mu, LPM_{0.9;3}), (\mu, \sigma)\) efficient frontiers in the \((\mu, \sigma)\) framework.

\(^9\) In Figure 13, note that the \(\mu/LPM(a=0.9)\) and the \(\text{UPM/LPM}(a=0.9, c=3)\) frontiers are identical.
When the graph uses the $\mu, \sigma$ axes, the mean-variance portfolio dominates. It should be clear that when mean-variance is utilized, the UPM/LPM portfolios are subsets of the mean-variance portfolios.

b. The Probability Distributions for $(a=0.9, c=0.5)$ portfolios.

Again, the return distributions were generated and a representative result is presented in Figure 12. The frequency of returns below the target increases in comparison with previous investment strategies, which corresponds with risk-seeking behavior in the downside part of distribution. On the other hand, the probability of the highest returns decreases because of the conservative upside potential strategy. The return distributions of the all portfolios have more area in the left tail than in the right tail (negative skewness). Compared to other models, the $(\mu, LPM_{0.9:3})$ portfolio distribution differs considerably in the upper part of the distribution, because this model does not assume potential upside aversion. Hence, the $(\mu, LPM)$ portfolios do not sufficiently express the current investor's wish not to take bets on high returns.
Figure 15 – Probability Distributions for \((UPM_{0.5;3}, LPM_{0.9;3}), (\mu, LPM_{0.9;3}), \text{ and } (\mu, \sigma)\) Portfolios.

Case 4: Downside Risk Seeking \((a=0.9)\) and Upside Potential Seeking \((c=3)\) Investment Strategy

An aggressive investment strategy is presented in this case. The investor wants to participate on the increasing markets whenever returns are above the minimum target return. Whenever returns are below-target, the main concern is not to fall short but without regard to the amount. Thus, our investor likes exposure to high returns and accepts exposure to low returns. In other words, the investor is upside potential seeking above the target return, and risk seeking below the target return. The utility function expressing these preferences is convex above the target return and approximately linear below the target. The slope of convexity usually changes at the target return because that is where the investor’s sensitivity to gains and losses changes. Again, we assume the degree of the risk seeking is slightly below risk neutrality \((a = 0.9)\), and degree of upside potential seeking is \(c = 3\) while the target return remains unchanged at 3%. Thus, the following utility function is generated:

\[
U(r) = \max\{(r-0.03) ; 0\}^2 -h\cdot\max\{(0.03-r) ; 0\}^{0.9}
\]  

(21)
Figure 16: Utility function for the Downside Risk Seeking (a=0.9) and Upside Potential Seeking (c=3) Investment Strategy.

**a. Efficient Frontiers for a=0.9, c=3.**

For comparison purposes, the \((UPM_{0.5;3}, LPM_{0.9;3})\) efficient frontier is computed in addition to the efficient frontiers computed for this case. In the \((\mu, \sigma)\) coordinate system, the \((UPM_{0.5;3}, LPM_{0.9;3})\) and \((UPM_{3;3}, LPM_{0.9;3})\) efficient frontiers are shifted further to the right than the \((\mu, LPM_{0.9;3})\) and \((\mu, \sigma)\) efficient frontiers. Thus, a higher level of standard deviation may be expected for the same level of portfolio return using the UPM/LPM methodology. Using the \((UPM_{3;3}, LPM_{0.9;3})\) coordinate system, the \((UPM_{3;3}, LPM_{0.9;3})\) efficient frontiers dominate. The other two LPM frontiers also dominate the \((\mu, \sigma)\) frontier.
Figure 17: \((U_{PM_{3;3}}, L_{PM_{0.9;3}}), (U_{PM_{0.5;3}}, L_{PM_{0.9;3}}), (\mu, L_{PM_{0.9;3}}), \text{and } (\mu, \sigma) \text{ Efficient Frontiers in the } (U_{PM_{3;3}}, L_{PM_{0.9;3}}) \text{ Coordinate System.}

Figure 18: \((U_{PM_{3;3}}, L_{PM_{0.9;3}}), (U_{PM_{0.5;3}}, L_{PM_{0.9;3}}), (\mu, L_{PM_{0.9;3}}), \text{and } (\mu, \sigma) \text{ Efficient Frontiers in the } (\mu, \sigma) \text{ Coordinate System.}
b. The Probability Distributions for $a=0.9, c=3$.

The $(UPM_{3,3}, LPM_{0.9,3})$ portfolio distribution exhibits high probability of returns far from the target and a low kurtosis. This agrees with the downside risk seeking and upside potential seeking behavior described in this case. The $(UPM_{3,3}, LPM_{0.9,3})$ optimal portfolio has a higher magnitude of above-target returns than the $(\mu, LPM_{0.9,3})$ optimal portfolios and the $(\mu, \sigma)$ optimized portfolios for the same level of downside risk. This results from $c>1$ in the calculation of $UPM$ making high upside deviations from the target return relatively more preferable than the lower ones. Only the $(UPM_{3,3}, LPM_{0.9,3})$ portfolios sufficiently carries out the investor’s wish to take bets on high returns.

![Figure 19: Probability Distributions for $(UPM_{3,3}, LPM_{0.9,3})$, $(\mu, LPM_{0.9,3})$, and $(\mu, \sigma)$ Portfolios.](image)
IV. Summary and Conclusion

The lower partial moment (LPM) has been the downside risk measure that is most commonly used in portfolio analysis. Its major disadvantage is that its inherent utility functions are linear above some target return. As a result, the upper partial moment (UPM)/lower partial moment (LPM) ratio was recently suggested by Sortino, van der Meer, and Plantinga (1999) as a method of dealing with investor utility above the target return. This paper proposes a general UPM/LPM portfolio model and has presented four utility case studies to illustrate its use.

The chief advantage of the general UPM/LPM model is that it encompasses a vast spectrum of utility theory. It includes the reverse S-shaped utility functions of Friedman and Savage (1948) and Markowitz (1952). It also includes the utility functions that are presented in Swalm (1966) and Fishburn (1977). Finally, the UPM/LPM model is consistent with the prospect theory utility functions proposed by Kahneman and Tversky (1979).
References


